

Dimensional Reduction in Non-Supersymmetric Theories

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ABSTRACT

It is shown that regularisation by dimensional reduction is a viable alternative to dimensional regularisation in non-supersymmetric theories.

1. Introduction

Dimensional regularisation (DREG) is an elegant and convenient way of dealing with the infinities that arise in quantum field theory beyond the tree approximation.^[1] It is well adapted to gauge theories because it preserves gauge invariance; it is less well-suited, however, for supersymmetry because invariance of an action with respect to supersymmetric transformations only holds in general for specific values of the space-time dimension d . This is essentially due to the fact that a necessary condition for supersymmetry is equality of Bose and Fermi degrees of freedom. In non-gauge theories it is relatively easy to circumvent this problem, and DREG as usually employed is, in fact, a supersymmetric procedure. Gauge theories are a different matter, however, and the question as to whether there exists a completely satisfactory supersymmetric regulator for gauge theories remains controversial. This fact has been exploited recently to suggest that there may be supersymmetric anomalies.^[2]

An elegant attempt to modify DREG so as to render it compatible with supersymmetry was made by Siegel.^[3] The essential difference between Siegel's method (DRED) and DREG is that the continuation from 4 to d dimensions is made by *compactification* or *dimensional reduction*. Thus while the momentum (or space-time) integrals are d -dimensional in the usual way, the number of field components remains unchanged and consequently supersymmetry is undisturbed. (For a pedagogical introduction to DRED see Ref. 4).

As pointed out by Siegel himself,^[5] there remain potential ambiguities with DRED associated with treatment of the Levi-Civita symbol, $\epsilon^{\mu\nu\rho\sigma}$. To see how these arise, recall that with DRED it would seem that necessarily $d < 4$, since the regulated action is, after all, defined by dimensional *reduction*. Then, given $d < 4$, one can define an object ϵ^{ijkl} (where all the indices are now d -dimensional) and show that algebraic inconsistencies result^[5] unless $d = 4$. A related problem (of course) is the fact that the only consistent treatment of γ^5 within DREG is predicated^[1] on having $d > 4$; for a discussion of this see Ref. 6, where it is suggested that one may in fact continue the relations we need to describe anomalies from $d > 4$ to $d < 4$ and so have our cake and eat it. There is another (but again related) problem with DRED, arising from the fact that in spite of the correct counting of degrees of freedom, there are still ambiguities associated with establishing invariance of the action. (Most apparent in component formalism: see Ref. 7). In spite of these reservations, DRED remains the regulator of choice for supersymmetric theories, and has survived practical tests to high loop levels.

In this paper, we address problems of a different nature, which arise when DRED is applied to non-supersymmetric theories. That DRED is a viable alternative to DREG was first claimed in Ref 4. Subsequently it has been adopted occasionally, motivated usually by the fact that Dirac matrix algebra is easier in four dimensions—and in particular by the desire to use Fierz identities. For a recent example, see Refs. 8–10. As we shall see, however, one must be very careful in applying DRED to non-supersymmetric theories because of the existence of *evanescent couplings*. These were first described in Ref. 11, and independently discovered by van Damme and 't Hooft in 1984.^[12] They argued, in fact, that while DRED is a satisfactory procedure for supersymmetric theories^[13,14] (modulo the subtleties alluded to above) it leads to a catastrophic loss of unitarity in the non-supersymmetric case. Evidently there is an important issue to be resolved here—is, as the authors of Ref. 12 claim, use of DRED forbidden (except in the supersymmetric case) in spite of its apparent convenience? We shall claim that if DRED is employed in the manner envisaged by the authors of Ref. 4, (which as we shall see differs in an important way from the definition of DRED primarily used in Ref. 12) then there is no problem with unitarity. We will present as evidence for this conclusion a set of transformations whereby the beta-functions of a particular theory (calculated using DRED) may be related to the beta-functions of the same theory (calculated using DREG) by means of coupling constant reparametrisation. The bad news is that a correct description of this (or any non-supersymmetric) theory impels us to a recognition of the fundamental fact that in general the evanescent couplings renormalise in a manner independent and different from the “ real ” couplings with which we may be tempted to associate them. This means that care must be taken as we go beyond one loop; nevertheless it is still possible to exploit the simplifications in the Dirac algebra which have motivated the use of DRED. We will return to this point later.

2. Gauge theory with fermions

We begin by considering a non-abelian gauge theory with fermions but no elementary scalars. The theory to be studied consists of a Yang-Mills multiplet $W_\mu^a(x)$ with a multiplet of spin $\frac{1}{2}$ Dirac[★] fields $\psi^\alpha(x)$ transforming according to an irreducible representation R of the gauge group G .

★ the generalisation to two component or Majorana fields is straightforward

The Lagrangian density (in terms of bare fields) is

$$L_B = -\frac{1}{4}G_{\mu\nu}^2 - \frac{1}{2\alpha}(\partial^\mu W_\mu)^2 + C^{a*}\partial^\mu D_\mu^{ab}C^b + i\bar{\psi}^\alpha \gamma^\mu D_\mu^{\alpha\beta}\psi^\beta \quad (2.1)$$

where

$$G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + gf^{abc}W_\mu^b W_\nu^c$$

and

$$D_\mu^{\alpha\beta} = \delta^{\alpha\beta}\partial_\mu - ig(R^a)^{\alpha\beta}W_\mu^a$$

and the usual covariant gauge fixing and ghost terms have been introduced.

The process of dimensional reduction consists of imposing that all field variables depend only on a subset of the total number of space-time dimensions- in this case d out of 4 where $d = 4 - \epsilon$. In order to fully appreciate the consequences of this procedure we must then make the decomposition

$$W_\mu^a(x^j) = \{W_i^a(x^j), W_\sigma^a(x^j)\} \quad (2.2)$$

where

$$\delta_i^i = \delta_j^j = d \quad \text{and} \quad \delta_{\sigma\sigma} = \epsilon.$$

It is then easy to show that

$$L_B = L_B^d + L_B^\epsilon \quad (2.3)$$

where

$$L_B^d = -\frac{1}{4}G_{ij}^2 - \frac{1}{2\alpha}(\partial^i W_i)^2 + C^{a*}\partial^i D_i^{ab}C^b + i\bar{\psi}^\alpha \gamma^i D_i^{\alpha\beta}\psi^\beta \quad (2.4)$$

and

$$L_B^\epsilon = \frac{1}{2}(D_i^{ab}W_\sigma^b)^2 - g\bar{\psi}\gamma_\sigma R^a\psi W_\sigma^a - \frac{1}{4}g^2 f^{abc}f^{ade}W_\sigma^b W_{\sigma'}^c W_\sigma^d W_{\sigma'}^e. \quad (2.5)$$

Conventional dimensional regularisation (DREG) amounts to using Eq. (2.4) and discarding Eq. (2.5). The additional contributions from L_B^ϵ are precisely what is required to restore the supersymmetric Ward identities at one loop in supersymmetric theories, as verified in Ref. 4.[†]

[†] Of course in simple applications it is in general more convenient to eschew the separation performed above and calculate with 4-dimensional and d -dimensional indices rather than d -dimensional and ϵ -dimensional ones.

We would now like to rewrite Eq. (2.4) and Eq. (2.5) in terms of renormalised quantities. This is usually done by simply rescaling all fields and coupling constants. It is clear, however, from the dimensionally reduced form of the gauge transformations:

$$\begin{aligned}\delta W_i^a &= \partial_i \Lambda^a + g f^{abc} W_i^b \Lambda^c \\ \delta W_\sigma^a &= g f^{abc} W_\sigma^b \Lambda^c \\ \delta \psi^\alpha &= ig (R^a)^{\alpha\beta} \psi^\beta \Lambda^a\end{aligned}\tag{2.6}$$

that each term in Eq. (2.5) is separately invariant under gauge transformations. The W_σ -fields behave exactly like scalar fields, and are hence known as ϵ -scalars. The significance of this is that there is therefore no reason to expect the $\bar{\psi}\psi W_\sigma$ vertex to renormalise in the same way as the $\bar{\psi}\psi W_i$ vertex (except in the case of supersymmetric theories). In the case of the quartic ϵ -scalar coupling the situation is more complex since in general of course more than one such coupling is permitted by Eq. (2.6). In other words, we cannot in general expect the $f - f$ tensor structure present in Eq. (2.5) to be preserved under renormalisation. This is clear from the abelian case, where there is no such quartic interaction in L_B^ϵ but there is a divergent contribution at one loop from a fermion loop.

At this point we have a choice. On the one hand, we could decide that it doesn't matter if Green's functions with external ϵ -scalars are divergent (since they are anyway unphysical) and introduce a common wave function subtraction for W_i and W_σ , a wave function subtraction for ψ and a coupling constant subtraction for g , these all being determined (as usual) by the requirement that Green's functions with real particles be rendered finite. This is the procedure adopted in the main by van Damme and 't Hooft. On the other hand we could insist on all Green's functions being finite, leading ineluctably to the introduction of a plethora of new subtractions or equivalently coupling constants. We argue strongly that it is only the latter procedure which has a chance of leading to a consistent theory; the former manifestly leads to a breakdown of unitarity (which in fact is the conclusion reached in Ref. 12). We proceed now to a discussion of the renormalisation of Eq.(2.3) : conducted in a somewhat old-fashioned way, in the interest (we hope) of clarity.

We are therefore led to consider the following expressions for renormalised quantities L^d and

L^ϵ :

$$\begin{aligned}
L^d = & -\frac{1}{4}Z^{WW}(\partial_i W_j - \partial_j W_i)^2 - \frac{1}{2\alpha}(\partial^i W_i)^2 \\
& - Z^{WWW}g f^{abc}\partial_i W_j^a W^{bi}W^{cj} - \frac{1}{4}Z^{4W}g^2 f^{abc}f^{ade}W_i^b W_j^c W^{di}W^{ej} \\
& + Z^{C^*C}\partial^i C^*\partial_i C + Z^{C^*CW}g f^{abc}\partial^i C^{a*}W_i^b C^c \\
& + Z^{\bar{\psi}\psi}i\bar{\psi}\gamma^i\partial_i\psi + Z^{\bar{\psi}\psi W}g\bar{\psi}R^a\gamma^i\psi W_i^a
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
L^\epsilon = & \frac{1}{2}Z^{\epsilon\epsilon}(\partial_i W_\sigma)^2 + Z^{\epsilon\epsilon W}g f^{abc}\partial_i W_\sigma^a W^{bi}W_\sigma^c \\
& + Z^{\epsilon\epsilon WW}g^2 f^{abc}f^{ade}W_i^b W_\sigma^c W^{di}W_\sigma^e - Z^{\bar{\psi}\psi\epsilon}h\bar{\psi}R^a\gamma^\sigma\psi W_\sigma^a \\
& - \frac{1}{4}\sum_{r=1}^p Z_r^{4\epsilon}\lambda_r H_r^{abcd}W_\sigma^a W_{\sigma'}^c W_\sigma^b W_{\sigma'}^d.
\end{aligned} \tag{2.8}$$

Eq. (2.7) is the usual expression for the Lagrangian in terms of renormalised parameters. The labelling of the various subtraction constants is not particularly conventional but (we hope) self-explanatory. In Eq. (2.8) we have introduced a ‘‘Yukawa’’ coupling h and a set of p quartic couplings λ_r . The number p is given by the number of independent rank four tensors H^{abcd} which are non-vanishing when symmetrised with respect to (ab) and (cd) interchange. In $SU(2)$, this number is 2: $\delta^{ab}\delta^{cd}$, and $\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}$. In $SU(3)$, $p = 3$, and for $SU(N)$ ($N > 3$), $p = 4$. The λ_r mix non-trivially under renormalisation; it is straightforward (but tedious) to calculate their one-loop β -functions in, for example, the $SU(N)$ case and check that for a supersymmetric theory the $f - f$ tensor structure is preserved. We will not tax the reader’s patience by presenting these results; but in the next section all the β -functions are calculated for a particular $SU(2)$ theory.

The results for some of the subtraction constants at one loop are as follows:

$$\begin{aligned}
Z^{WW} &= 1 + \frac{1}{16\pi^2\epsilon}g^2[(\frac{13}{3} - \alpha)C_2(G) - \frac{8}{3}T(R)] \\
Z^{WWW} &= 1 + \frac{1}{16\pi^2\epsilon}g^2[(\frac{17}{6} - \frac{3}{2}\alpha)C_2(G) - \frac{8}{3}T(R)] \\
Z^{\bar{\psi}\psi} &= 1 - \frac{1}{16\pi^2\epsilon}g^2[2\alpha C_2(R)] \\
Z^{\bar{\psi}\psi W} &= 1 - \frac{1}{16\pi^2\epsilon}g^2[\frac{3+\alpha}{2}C_2(G) + 2\alpha C_2(R)]
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
Z^{\epsilon\epsilon} &= 1 + \frac{1}{16\pi^2\epsilon} [g^2(6 - 2\alpha)C_2(G) - 4h^2T(R)] \\
Z^{\epsilon\epsilon W} &= 1 + \frac{1}{16\pi^2\epsilon} [g^2\frac{9 - 5\alpha}{2}C_2(G) - 4h^2T(R)] \\
Z^{\bar{\psi}\psi\epsilon} &= 1 - \frac{1}{16\pi^2\epsilon} g^2[(6 + 2\alpha)C_2(R) - (3 - \alpha)C_2(G)] - \frac{1}{16\pi^2\epsilon} h^2[2C_2(G) - 4C_2(R)]
\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
\delta^{ab}C_2(G) &= f^{acd}f^{bcd} \\
\delta^{ab}T(R) &= \text{Trace}[R^a R^b] \quad \text{and} \\
C_2(R)1 &= R^a R^a.
\end{aligned}$$

At one loop, subtraction constants with no external ϵ -scalars depend only on the real couplings while subtraction constants with external ϵ -scalars depend on both real and evanescent couplings. Notice that we are using minimal subtraction, so that all the Z 's contain only poles in ϵ . From Eqs. (2.9) and (2.10) it is easy to verify Slavnov-Taylor identities such as :

$$\frac{Z^{WWWW}}{Z^{WW}} = \frac{Z^{\epsilon\epsilon W}}{Z^{\epsilon\epsilon}} = Z^g \sqrt{Z^{WW}} \tag{2.11}$$

where Z^g is the renormalisation constant for g .

It is straightforward to calculate the one loop β -function for h ; the result is :

$$\beta_h(g, h) = \frac{h}{16\pi^2} [(4h^2 - 6g^2)C_2(R) + 2h^2T(R) - 2h^2C_2(G)] \tag{2.12}$$

which is to be contrasted with the result for β_g , which is just as for DREG:

$$\beta_g(g) = \frac{g^3}{16\pi^2} [\frac{4}{3}T(R) - \frac{11}{3}C_2(G)]. \tag{2.13}$$

The fact that β_g is independent of h at one loop is a trivial consequence of minimal subtraction. It is interesting that this remains true at two loops[★]: moreover the result for $\beta_g^{(2)}$ is precisely the same as that obtained using DREG. We do not know whether this persists to all orders. In section (3), however, we will find that in a more general theory (specifically one involving genuine scalar particles) there *are* real couplings whose β -functions depend on evanescent couplings beyond one loop.

★ this is clear from the calculations presented in Ref. 4, although not emphasised because the distinction between g and h was not made in that paper.

We see that even if we choose to set $h = g$, the two β -functions are not identical, unless we also have that

$$5C_2(G) + 2T(R) = 6C_2(R). \quad (2.14)$$

The solution $C_2(G) = C_2(R) = 2T(R)$ corresponds to a supersymmetric theory. (The awkward factor of 2 in front of $T(R)$ is due to our adoption of Dirac fermions). In general it is clear that the choice $h = g$ is not renormalisation group invariant: that is, if it is made at one renormalisation scale it is not true at another. Hence to investigate the structure of the renormalised theory it is important that this choice (which has been implicitly made in most DRED calculations) be *not* made. We will return later to the issue of the validity of calculations involving $h = g$ and corresponding choices for the other evanescent couplings.

It is well known that different regulation procedures lead in general to different results for (amongst other things) β -functions. We may expect, however, that the β -functions obtained with two different regulators can be transformed into each other by means of coupling constant redefinition. It was this procedure which, for example, established the equivalence of the DRED and DREG results for the $N = 4$ supersymmetric gauge theory. In Ref. 12 it is asserted that it is *not* possible to transform the DRED β -functions into the DREG ones; both in general and in the context of a particular model (which we call the DH model). In the next section, we show how in fact there does exist such a transformation, as long as we both implement DRED in the manner described above and maintain the distinction between real and evanescent couplings. On the basis of this result we intend to argue that DRED is quite valid and (contrary to Ref 12) a perfectly valid alternative to DREG in the non-supersymmetric case.

3. The DH Model

In this Section we show explicitly how the β -functions evaluated using our version of DRED may be transformed into the β -functions evaluated with DREG, with reference to a specific example—namely the toy model introduced in Ref. 12. As mentioned earlier, the version of DRED used in Ref. 12 is crucially different from the one which we advocate, and in fact leads to β -functions which are not equivalent up to coupling constant redefinition to those of DREG. We shall, however, maintain that our implementation of DRED is natural and appropriate.

The toy model considered by van Damme and 't Hooft in Ref. 12 has gauge group $SU(2)$, a multiplet of Dirac fields ψ and a multiplet of scalar fields ϕ each of which transforms according to the adjoint representation. The bare Lagrangian is

$$L = L_B^d + L_B^\epsilon \quad (3.1)$$

where

$$\begin{aligned} L_B^d = & -\frac{1}{4}(G_{ij}^a)^2 + \frac{1}{2}(D_i\phi^a)^2 + i\bar{\psi}^a \not{D}\psi^a \\ & + iy\epsilon^{abc}\bar{\psi}^a\phi^b\psi^c - \frac{1}{8}\lambda(\phi^2)^2 \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} L_B^\epsilon = & +\frac{1}{2}(D_i W_\sigma^a)^2 - \frac{1}{2}\rho_1(W_\sigma^a)^2\phi^2 + \frac{1}{2}\rho_2(W_\sigma^a\phi^a)^2 \\ & + ih\epsilon^{abc}\bar{\psi}^a W_\sigma^b\gamma_\sigma\psi^c - \frac{1}{4}\rho_4(W_\sigma^2)^2 + \frac{1}{4}\rho_5(W_\sigma^a W_\tau^a)^2. \end{aligned} \quad (3.3)$$

(We have omitted ghost and gauge fixing terms). The corresponding renormalised Lagrangian $L = L^d + L^\epsilon$ is obtained in an analogous fashion to that for the gauge theory with fermions in Section (2). In particular, we have

$$y \rightarrow Z^y y, \quad \lambda \rightarrow Z^\lambda \lambda, \quad h \rightarrow Z^h h, \quad \rho_i \rightarrow Z^{\rho_i} \rho_i, \quad i = 1, 2, 4, 5. \quad (3.4)$$

The one-loop β -functions for the various couplings, both real and evanescent, can be calculated by standard methods; the results are:

$$\begin{aligned}
\beta_\lambda^{(1)} &= 11\lambda^2 - 24\lambda\tilde{g} + 24\tilde{g}^2 + 16\lambda y - 32\tilde{y}^2 \\
\beta_{\tilde{y}}^{(1)} &= 16\tilde{y}^2 - 24\tilde{y}\tilde{g} \\
\beta_{\tilde{g}}^{(1)} &= -\frac{26}{3}\tilde{g}^2 \\
\beta_{\rho_1}^{(1)} &= 8\rho_1^2 + 2\rho_2^2 + 4\rho_1\rho_4 - 8\rho_1\rho_5 + 2\rho_2\rho_5 \\
&\quad - 16\rho_3\tilde{y} + 4\tilde{g}^2 + \rho_1(-24\tilde{g} + 8\tilde{y} + 8\rho_3) + 5\lambda\rho_1 - \lambda\rho_2 \\
\beta_{\rho_2}^{(1)} &= -10\rho_2^2 + 16\rho_1\rho_2 + 4\rho_2\rho_4 - 2\rho_2\rho_5 - 6\tilde{g}^2 \\
&\quad + \rho_2(-24\tilde{g} + 8\tilde{y} + 8\rho_3) + 2\lambda\rho_2 \\
\beta_{\rho_3}^{(1)} &= \rho_3(-12\tilde{g} + 16\rho_3) \\
\beta_{\rho_4}^{(1)} &= 16\rho_4^2 + 6\rho_5^2 - 16\rho_4\rho_5 - 16\rho_3^2 + 6\tilde{g}^2 \\
&\quad + \rho_4(-24\tilde{g} + 16\rho_3) + 6\rho_1^2 - 4\rho_1\rho_2 \\
\beta_{\rho_5}^{(1)} &= -14\rho_5^2 + 24\rho_4\rho_5 - 6\tilde{g}^2 + \rho_4(-24\tilde{g} + 16\rho_3) - 2\rho_2^2
\end{aligned} \tag{3.5}$$

where we have defined $\tilde{y} = y^2$, $\tilde{g} = g^2$ and $\rho_3 = h^2$. Here and subsequently we suppress a factor of $(16\pi^2)^{-L}$ in the expression for an L -loop β -function contribution. Note that, as in section (2), the one-loop β -functions for real couplings do not depend on the evanescent couplings due to the extra factor of ϵ associated with the ϵ -scalars, a fact which will simplify the consideration of coupling constant redefinitions later.

We now want to compare the DRED and DREG results for the two-loop beta functions. At the two-loop level, we noted in section (2) that, for the class of theories considered there, the DRED result for β_g was independent of the evanescent couplings; this property persists for the DH model and is clearly true in general. Let us now consider the DRED calculation of $\beta_\lambda^{(2)}$ and $\beta_{\tilde{y}}^{(2)}$. There are two classes of two-loop diagrams contributing to the renormalisation of real couplings; those which involve ϵ -scalars and those which do not. The set of diagrams which do not involve ϵ -scalars of course yield the same result as for dimensional regularisation. In other words the difference between DREG and DRED arises solely from the graphs with ϵ -scalars, and consequently we shall limit our attention to these. The calculation of the contributions to the β -functions from this class of graphs is rather straightforward, since the presence of a factor of ϵ from the multiplicity of the ϵ -scalars means that we only need the double pole from the Feynman integral. Each graph also requires corresponding counter-term diagrams, and it is perhaps appropriate at this point to explain the difference between our prescription for DRED, and that adopted in Ref. 12, since the difference resides in our treatment of the counter-terms. According to our prescription,

we construct the counter-term diagrams for a graph with an ϵ -scalar loop on its own merits, by replacing divergent sub-diagrams of the original diagram by counter-term insertions with the same pole structure. This is equivalent to constructing one-loop counter-term diagrams with insertions of $Z^{\lambda(1)}$, $Z^{\tilde{g}(1)}$, $Z^{\tilde{y}(1)}$, $Z^{\rho_i(1)}$. van Damme and 't Hooft , on the other hand, construct one-loop counter-term diagrams by replacing counter-terms for evanescent couplings by those for the corresponding real couplings, i.e. replacing $Z^{\rho_i(1)}$ by $Z^{\tilde{g}(1)}$. This prescription certainly seems incompatible with our philosophy of taking the evanescent couplings seriously and distinguishing them from real couplings, since it would not eradicate divergences from graphs with external ϵ -scalars. With our prescription, however, some care needs to be exercised in determining the correct counter-term diagrams, since a sub-loop with ϵ -scalars, external real fields and a divergent Feynman integral is nevertheless finite owing to the multiplicity-factor of ϵ , and hence does not require a subtraction.

The set of graphs to be calculated may be easily obtained from Refs. 15- 17, by replacing one or more vector propagators by ϵ -scalar ones. The fact that there is no $W_\sigma - \phi - \phi$ vertex is a considerable simplification. The results for the *difference* between DRED and DREG calculations (i.e. $\delta\beta = \beta_{DRED} - \beta_{DREG}$) are as follows:

$$\begin{aligned}
\delta\beta_{\tilde{g}}^{(2)} &= 0 \\
\delta\beta_{\tilde{y}}^{(2)} &= 16\tilde{g}^2\tilde{y} \\
\delta\beta_{\lambda}^{(2)} &= -4\lambda(36\rho_1^2 + 16\rho_2^2 - 24\rho_1\rho_2) + 8(24\rho_1^3 - 12\rho_2^3 \\
&\quad + 32\rho_1\rho_2^2 - 24\rho_1^2\rho_2) + 32\rho_4(3\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2) \\
&\quad - 32\rho_5(6\rho_1^2 + \rho_2^2 - 4\rho_1\rho_2) - 96(3\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2)\tilde{g} \\
&\quad + 96(2\rho_1 - \rho_2)\tilde{g}^2 - 16\tilde{g}^3 - 128(3\rho_1 - \rho_2)\rho_3\tilde{y} \\
&\quad + 16\rho_3(12\rho_1^2 - 8\rho_1\rho_2 + 4\rho_2^2) + 16\lambda\tilde{g}^2.
\end{aligned} \tag{3.6}$$

The fact that $\delta\beta_{\tilde{y}}^{(2)}$ is also independent of the evanescent couplings is quite remarkable, resulting from a cancellation of a large number of individual contributions. By itself, this result would have lent support to the conjecture^[11] that the real coupling DRED β -functions are indeed independent of the evanescent couplings. The conjecture is, however, laid to rest by $\delta\beta_{\lambda}^{(2)}$. We now proceed to show that the DRED results are nevertheless equivalent to the DREG ones in the sense that they may be transformed into them by a finite perturbative reparametrisation of the coupling constants.

Given a theory with coupling constants $\{\lambda_i\}$, if we define new couplings $\{\lambda_i'\}$ by $\lambda_i' = \lambda_i + \delta\lambda_i$ then the resulting change in the two loop β -functions is given by

$$\delta\beta_i = \beta_i^{(2)'}(\lambda) - \beta_i^{(2)}(\lambda) = \beta_j^{(1)} \frac{\partial}{\partial\lambda_j} \delta\lambda_i - \delta\lambda_j \frac{\partial}{\partial\lambda_j} \beta_i^{(1)}. \quad (3.7)$$

Our task is to demonstrate a set of $\delta\lambda_i$ for the DH model such that the resulting $\delta\beta_i$ precisely cancel Eq. (3.6), restoring the DREG results for the real β functions. This is a straightforward calculation (the tedium of which was ameliorated by employment of REDUCE) and the result is that the following set of $\delta\lambda_i$ does the business:

$$\begin{aligned} \delta\lambda &= \frac{1}{26} \Delta (11\lambda^2 + 24\tilde{g}^2 - 32\tilde{y}^2 - 24\lambda\tilde{g} + 16\lambda\tilde{y}) + 12\rho_1^2 + 4\rho_2^2 - 8\rho_1\rho_2 \\ \delta\tilde{y} &= \frac{1}{26} \Delta (16\tilde{y}^2 - 24\tilde{g}\tilde{y}) \\ \delta\tilde{g} &= \frac{1}{3} (2 - \Delta) \tilde{g}^2 \end{aligned} \quad (3.8)$$

where Δ is an arbitrary constant. It should be emphasised that this is a non-trivial result in that the existence of the solution *does* depend on the precise values of the coefficients in Eq. (3.6).

It is clear that the required transformations become particularly simple if we take $\Delta = 0$ in Eq. (3.8). The resulting $\delta\lambda$ and $\delta\tilde{g}$ then correspond precisely to the potential finite contributions to $Z^\lambda\lambda$ and $Z^{\tilde{g}}\tilde{g}$ which arise at one loop when a divergence from a Feynman integral is multiplied by a multiplicity factor of ϵ . (Of course, since we are using minimal subtraction prescription we discard these finite contributions.) This is intriguing, since a prescription mentioned *en passant* in Ref. 12 (their “system 3”) also led (for a general theory) to β -functions related to those for DRED by the same transformation. This prescription is not manifestly identical to ours, since the multiplicity factor for the ϵ -scalars is set to ϵ only at the end of the calculation—in other words, from our point of view the counter-terms are not evaluated using minimal subtraction; nevertheless we see from the above remarks that it leads to identical results, at least for the DH model. It would be interesting to prove the equivalence of these two prescriptions in general.

4. Discussion

The dependence on the evanescent couplings of the DRED β -functions seems to pose a serious problem. Most DRED users have in fact dealt with the evanescent couplings by setting them equal to the real couplings suggested by the bare Lagrangian; in the DH model, for example, this corresponds to $\rho_i = \tilde{g}$ for all i . But the evanescent couplings evolve differently, so it would seem that two “observers” testing physics at different energy scales could not both make this choice. We saw in the last section, however, that (at least for the DH model through two loops) the DRED β -functions for the real couplings are in fact equivalent to the DREG ones. Of course if DRED and DREG are really to describe the same physics, it is important that the coupling constant redefinition that achieves this also renders the DRED S-matrix (for real particles) identical to that for DREG - in particular, independent of the evanescent couplings. A simple example of this effect in the DH model is as follows. Consider the one loop contributions to the ϕ^4 interaction from a pair of ρ_1 and/or ρ_2 vertices. These generate finite contributions to the vertex and so it is quite clear that with DRED the cross-section for $\phi - \phi$ scattering, for example, depends on ρ_1 and ρ_2 . But it is easy to check that the redefinition of λ from Eq. (3.8) is precisely what is required to remove these contributions. So when we start with DRED and implement Eq. (3.8) not only are the resulting β -functions the same as the DREG ones, but the resulting S-matrix is also identical. The evanescent sector is completely decoupled, and goes away as $\epsilon \rightarrow 0$, because we have been careful to make Green’s functions with external scalars finite.

Now DRED is only really useful in non-supersymmetric theories if we can calculate without splitting off the ϵ -scalars, so that Dirac algebra can be carried out in four dimensions. This involves setting the evanescent couplings equal to their “natural” values, as we described above. It should now be clear, however, that this procedure is quite harmless, as long as sub-divergences are subtracted at the level of the integrals, instead of by means of counter-term insertions. Great care would be needed however, if comparison between calculations performed with different values of the renormalisation scale μ were to be desired.

In conclusion, we have argued that DRED as it has been customarily employed is perfectly valid. We have demonstrated this through two loops in a simple model; but it seems clear that the conclusions are general: since both DRED and DREG start from the same bare theory, it is inevitable that the corresponding renormalised theories are equivalent physically, as indeed we find. Nevertheless, we feel that it is important that users of DRED be aware of the existence and

significance of the evanescent couplings.

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